Some remarks on the theorems of Gleason and Kochen-Specker

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Abstract

A Gleason-type theorem is proved for two restricted classes of informationally complete POVMs in the qubit case.

A particular (incomplete) Kochen-Specker colouring, suggested by Appleby in dimension three, is generalized to arbitrary dimension. We investigate its effectivity as a function of dimension, using two different measures of this. In particular, we will derive a limit for the fraction of the sphere that can be satisfactorily coloured using the generalized Appleby construction as the number of dimensions approaches infinity. The second, and physically more relevant measure of effectivity, is to look at the fraction of possible ON-bases properly coloured. Using this measure, we will derive a 'lower bound for the upper bound' in three and four real dimensions.

1 Introduction

Gleason's theorem [1] is a central result in mathematical physics. From it can be derived the 'quantum rule' for calculating probabilites, by taking the trace of the product between (the matrix representation of) the relevant projection operator and the so-called density matrix. The effective statement of the theorem, as applied to quantum physics, is that the density matrix is the unique form in which probabilites can be introduced into quantum mechanics. Let the function f associate to every subspace of some Hilbert space a number between 0 and 1 such that

$$\sum_{i=1}^{D} f(P_i) = 1 \tag{1}$$

where the P_i are the projection operators associated to the vectors $\mid e_i\rangle$, which form an ON-basis for the Hilbert space, and D is the dimension of the space. This assumption about f is enough to lead us to the concept of the density matrix. The proof of Gleason's theorem given this assumption is valid only for Hilbert spaces of dimension at least three, but requires a detour through two-dimensional subspaces.[2]

In a 2003 paper, Busch [5] makes a somewhat stronger assumption about this so-called frame function f, namely that

$$\sum_{i=1}^{N} f(E_i) = 1 \tag{2}$$

for all positive operators E_i summing to the identity. Note that N here may well be larger than D. The projection operators of Gleason's original proof are a special case of such operators, with $N = D$ and with the restriction that the operators be mutually orthogonal. With this assumption, a Gleason-type result is valid already in two dimensions, and the proof is considerably simplified. This theorem, analoguous to that of Gleason was later proved by Caves et al [6] by slightly different means.

In the first section of this paper, we will position ourselves between Gleason and Busch/Caves et al. For continuous frame functions, Gleason's theorem is valid in dimension two with the assumptions of the latter but not the former. The ambition here will be to look at sets of operators somewhat more general than those of Gleason, although more restricted than those of Caves et al. A strongly restricted set (also considered by Caves et al) is the set of sic-POVMs. Recall that in the qubit case, just as a PVM can be interpreted as two antipodal points on the Bloch sphere, a four element POVM can be seen as a tetrahedron inscribed in the Bloch sphere. In the case of sic-POVMs, this tetrahedron is regular. This set is not sufficient to enforce the Gleason result. The POVMs dealt with in this paper are generic four-element POVMs, irregular tetrahedrons with vertices on the Bloch sphere. We will show that, assuming continuity, a Gleason-type theorem can be proved for all members of two restricted classes of informationally complete POVMs in the qubit case, with the exception of the sic-POVM. We're thereby making likely that the same result holds for all generic asymmetric ic- POVMs. The sic-POVM, the symmetric variety of a four element POVM, has earlier been shown by Caves et al not to enforce the quantum probability rule in the qubit case. The method that will be used here is largely similar to theirs.

A theorem somewhat more well-known than Gleason's is that of Kochen and Specker [3], [4]. In the notation introduced above, KS deals with frame functions f from the set of projection operators to the set $\{0, 1\}$ satisfying the sum condition (1). The statement of the theorem is that no such truth value assignments are possible. This result is often translated in terms of colourings of spheres, and the same terminology will be used here. The effective statement of KS (in D real dimensions) is in these terms that no complete colouring of the D-sphere obeying the KS criteria is possible. The question then arises how close to complete we can possibly get. In a 2003 paper, Appleby [9] considers this question in the case of three-dimensional real Hilbert spaces, and suggests an incomplete colouring that provides a lower bound for the maximal effectivity in terms of area of S^2 coloured. In the second section of this paper, this particular colouring is generalized to arbitrary dimension. We investigate its effectivity as a function of dimension, using two different measures of this. In particular, we will show that the fraction of the sphere that can be satisfactorily coloured using the generalized Appleby construction does not go to zero as the number of dimensions approaches infinity, contrary to what one might have expected. The second, and physically more relevant measure of effectivity, is to look at the fraction of possible ON-bases properly coloured. Using this measure, we will derive a 'lower bound for the upper bound' in three and four real dimensions.

2 Gleason for the deformed sic-POVM

In what follows, we will investigate whether a Gleason-type theorem can be proved for two restricted classes of POVMs, namely two 'semi-symmetrical' families of asymmetrical POVMs with four elements, in the qubit case. Recall the formal definition of a POVM: a set of N positive operators E_i that act on a D dimensional Hilbert space $\mathscr{H}^D$, and that satisfy

$$\sum_{i=1}^{N} E_i = 1, \quad E_i^\dagger = E_i, \quad E_i \geq 0, \quad i = 0, 1, ..., D \tag{3}$$

Note in particular that the number of elements of a POVM need not equal the dimension of the Hilbert space. Both PVMs (Projective Valued Measures - in effect, ON bases; orthogonal resolutions of the identity) and POVMs can be said to represent measurements, but the POVM is a realization of a more general notion of measurement. For a PVM, the results obtained are mutually exclusive (for instance, $m = j$ means $m \neq j-1, ..., -j$). For a POVM, however, this is not the case. Also, while the projectors of a PVM always commute, no such assumption is made for the effects of a general POVM.

In what follows, the frame function will be assumed to be continuous, which allows for an expansion in terms of spherical harmonics.

The general two-dimensional effect can be expressed in terms of the Pauli matrices as

$$E = r1 + s \cdot \sigma = r1 + s\hat{n} \cdot \sigma \tag{4}$$

where 1 is the unit matrix and $\hat{n}$ is a unit vector.

The restricted sets of POVMs considered in this section all consist of effects that are multiples of one-dimensional projectors so that $r = s \leq \frac{1}{2}$ and $E = r(1 + \hat{n} \cdot \sigma)$. All effects also have the same weight r, namely $\frac{1}{N}$ for a N outcome POVM. A POVM is then fully specified by the $\hat{n}$ vectors of its elements, that sum to zero in order for the effects to sum to one:

$$\sum_{j=1}^{N} \hat{n}_j = 0 \tag{5}$$

for any POVM with N elements. All POVMs that are the same up to a three-dimensional rotation are considered equivalent.

We will now be interested in frame functions, i.e. probability distributions, $\tilde{f}(\frac{1}{N}(1 + \hat{n} \cdot \sigma)) \equiv f(\hat{n})$ defined on this set. From the right hand side of this equation it is clear that f is a function on the unit sphere in three dimensions. The normalization

$$\sum_{i=1}^{N} f(\hat{n}_i) = 1 \tag{6}$$

accounts for the fact that the probability of obtaining some outcome is one. Proving a Gleason-type theorem is equivalent to showing that a continuous frame function satisfying (6) has to be of the form

$$f(\hat{n}) = \mathrm{Tr}(\rho E) = \frac{1}{N}(1 + \hat{n} \cdot P) \tag{7}$$

in agreement with the standard quantum rule. Here, ρ is a positive operator of unit trace and $P = \mathrm{Tr}(\rho\sigma)$ is a three component vector satisfying $\mid P \mid \leq 1$. The frame function is a function on the unit sphere and can be written as a sum of spherical harmonics Y_{lm};

$$f(\hat{n}) = \sum_{lm} c_{lm} Y_{lm} \tag{8}$$

The quantum rule for the frame function allows only harmonics with $l = 0$ and $l = 1$; explicitly

$$\frac{1}{N}(1 + \hat{n} \cdot P) = \sqrt{\frac{4\pi}{N}} Y_{00} + \sqrt{\frac{2\pi}{3N}} P_x (Y_{1,-1} - Y_{1,1}) + i\sqrt{\frac{2\pi}{3}} P_y (Y_{1,-1} + Y_{1,1}) + \sqrt{\frac{4\pi}{3N}} P_z Y_{10} \tag{9}$$

Hence, we will want to expand the frame function over the particular POVM we are looking at in terms of spherical harmonics, and check what values of l contribute to the sum.
It can be shown that if the lth harmonic is to be allowed in the expansion of a frame function $f(\hat{n})$ $c_{lm} \sum_{j=1}^{N} Y_{lr}(\hat{n}_j) = \delta_{l0}$ has to hold for all l, m and r.
For $l = 0$ this condition is trivial, and is accomplished simply by normalization.
For $l \geq 1$ it is equivalent to either

$$c_{lm} = 0, \quad m = -l, ..., l \tag{10}$$

or

$$\sum_{j=1}^{N} Y_{lr}(\hat{n}_j) = 0, \quad r = -l, ..., l \tag{11}$$

This means that if the lth harmonic contributes to a frame function $f(\hat{n})$ that is, if not all of the c_{lm} are equal to zero, then the $\hat{n}$ vectors of the POVM must satisfy equation (11).
If we can show that for a certain POVM no l-values other than $l = 0$ and $l = 1$ can satisfy this for all m, we will have derived the quantum rule, thereby proving a Gleason-type theorem, because the fact that the frame function is real requires that the harmonics appear in exactly the combinations of equation (9).
Due to the property of the POVM $\hat{n}$-vectors (5), equation (11) is automatically satisfied for $l = 1$.
In investigating whether higher values of l can contribute we will make use of some properties of the spherical harmonics, namely the way that the θ and ϕ, or for cartesian coordinates $(\hat{n})_z$ and $(\hat{n})_x + i(\hat{n})_y$, dependencies can be separated, according to

$$Y_{lm}(\hat{n}) = Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \mathrm{e}^{im\phi} = \tag{12}$$

$$= h_{lm}((\hat{n})_z)((\hat{n})_x + i(\hat{n})_y)^m \tag{13}$$

where

$$h_{lm}((\hat{n})_z) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} \frac{P_l^m((\hat{n})_z)}{(\sqrt{1-(\hat{n})_z^2})^m} \tag{14}$$

and also, how the spherical harmonics transform under reflection, parity and conjugation

$$Y_{lm}(\hat{n}) = (-1)^{l+m} Y_{lm}(\pi - \theta, \phi) = (-1)^m Y_{lm}(\theta, \phi + \pi) = (-1)^l Y_{lm}(-\hat{n}) = (-1)^m Y_{l-m}^*(\hat{n}) \tag{15}$$

A particularly useful form of equation (13) is that for $m = l$, in which case

$$Y_{lm}(\hat{n}) = Y_{ll}(\hat{n}) \propto ((\hat{n})_x + i(\hat{n})_y)^l \tag{16}$$

The function $h_{lm}((\hat{n})_z)$ will in this case be independent of $(\hat{n})_z$, so that

$$\sum_{j=1}^{4} Y_{ll}(\hat{n}) \propto \sum_{j=1}^{4} ((\hat{n}_j)_x + i(\hat{n}_j)_y)^l \tag{17}$$

We will also make use of this 1984 result due to N.H.J. Lacroix [8]: *Any two non-zero associated Legendre functions $P_n^m(z)$ and $P_n^s(z)$, where n is an integer such that $n \geq 1$ and $m \neq \pm s$, have on the open interval $(-1,1)$ either no common zero or exactly one common zero. The latter occurs if and only if $n- \mid m \mid$ and $n- \mid s \mid$ are both odd and positive.*

For a POVM with four elements, the vectors $\{\hat{n}_i\}$ specify the vertices of a tetrahedron. The unit vectors of the sic-POVM considered by Caves et al form a regular tetrahedron. If this tetrahedron is stretched, using a parameter θ, a class of POVMs is obtained. In this first case, we will choose the θ dependence so that $\cos\theta = 0$ and $\cos\theta = \pm 1$, correspond to the square and the line, respectively. All values in between correspond to a unique four element POVM.

The four unit vectors specifying a POVM in this class of deformations of the sic-POVM can be expressed as

$$\begin{array}{rcccl} \hat{n}_1 & = & (\sin\theta\cos\frac{3\pi}{2}, \sin\theta\sin\frac{3\pi}{2}, \cos\theta) & = & (0, -\sin\theta, \cos\theta) \\ \hat{n}_2 & = & (\sin\theta\cos\frac{\pi}{2}, \sin\theta\sin\frac{\pi}{2}, \cos\theta) & = & (0, \sin\theta, \cos\theta) \\ \hat{n}_3 & = & (\sin\theta - \pi\cos 0, \sin\theta - \pi\sin 0, \cos\pi - \theta) & = & (\sin\theta, 0, -\cos\theta) \\ \hat{n}_4 & = & (\sin\theta - \pi\cos\pi, \sin\theta - \pi\sin\pi, \cos\pi - \theta) & = & (-\sin(\theta), 0, -\cos\theta) \end{array} \tag{18}$$

It is easily verified that the vectors sum to zero. The regular tetrahedron corresponds to

$$\theta_{sym} = \arccos\frac{1}{\sqrt{3}} \tag{19}$$

To investigate what harmonics can contribute, we start by looking at the sum

$$\sum_{j=1}^{4} Y_{ll}(\hat{n}_j) \propto \sum_{j=1}^{4}((\hat{n}_j)_x + i(\hat{n}_j)_y) \tag{20}$$

by equation (17). Using (18),

$$\sum_{j=1}^{4} Y_{ll}(\hat{n}_j) \propto 1^l + (-1)^l + (i)^l + (-i)^l \tag{21}$$

We see that this sum is zero for $l = 0, 1, 2$ and odds. To find out if harmonics with these values of l actually contribute, we have to check whether the sum $\sum_{j=1}^{4} Y_{lm}(\hat{n}_j)$ is zero for all m, not only for $m = l$. For $l = 2$, the sum will be zero for all m and all θ except for $m = 0$. However, for the values of θ that are solutions of $P_2^0(\cos\theta) = 0$ the $l = 2$ harmonic will contribute to the sum, and the quantum rule will not hold.

The zeros of $P_2^0(\cos\theta)$ are $\cos\theta = \pm 1, 0, \pm\frac{1}{\sqrt{3}}$; that is, the line, the square and the regular tetrahedron. The line and the square do not correspond to informationally complete POVMs. For $l = 3$ the only sum which does not give zero for all values of θ is $\sum_{j=1}^{4} P_3^2(\cos\theta)$, the zeros of which are $\cos\theta = 0$ and ± 1.

Also for $l = 5$, the sum for $l = 2$ differs from zero for all θ but those satisfying $P_5^2(\cos\theta) = 0$. The zeros are the same as those for $P_3^2(\cos\theta)$. This is consistent with the results of Caves et al.

For any odd l, $l + 2$ will be odd, so that $P_l^m(\cos(\pi - \theta)) = -P_l^m(\cos\theta)$ by equation (15). Also, the ϕ dependence is periodic with period 4π. This means that $\sum_{j=1}^{4} P_l^m(\cos\theta)$ will be non-zero for all odd l and $m \equiv 2 \pmod 4$. The condition for the l:th harmonic to contribute is that the sum $\sum_{j=1}^{4} Y_{lm}(\hat{n}_j)$ is zero for all m.

For odd $l \geq 7$ (even values of l have already been excluded) only those values of θ can contribute that are zeros of both $P_l^2(\cos\theta)$, $P_l^6(\cos\theta)$ and so on all the way up to $m = l - 1$.

Hence, in order for harmonics with $l \geq 7$, for which at least $\sum_{j=1}^{4} Y_{l2}(\hat{n}_j)$ and $\sum_{j=1}^{4} Y_{l6}(\hat{n}_j)$ are non-zero, to contribute for specific values θ_0 of θ we need these θ_0 to be common zeros of the l:th associated Legendre functions $P_l^m(\cos\theta)$ for different m. By the above theorem no such common

zeros of the associated Legendre functions exist, other than $\cos\theta = \pm 1$ and 0.
So, we have found that for this family of tetrahedra, we have a Gleason-type theorem (meaning that the quantum rule for calculating probabilites is valid and unique) for all configurations except for the (lower-dimensional) extreme points and the regular tetrahedron representing the sic-POVM.
The same reasoning can be applied to another family of POVMs, with similar semi-regular properties. This time, we choose the deformation parameter θ so that the unit vectors of this class of POVMs span tetrahedra whose base is always a regular triangle (in the extremal point, one of the vectors is the zero vector, and the configuration is just a trine).
This subset of tetrahedra can be parametrized as

$$\begin{array}{rcll} \hat{n}_1 & = & (0,0,-3\cos\theta\cos 0) & = (0,0,-3\cos\theta) \\ \hat{n}_2 & = & (\sin\theta\cos\frac{2\pi}{3},\sin\theta\sin\frac{2\pi}{3},\cos\theta) & = (-\frac{1}{2}\sin\theta,\frac{\sqrt{3}}{2}\sin\theta,\cos\theta) \\ \hat{n}_3 & = & (\sin\theta\cos\frac{4\pi}{3},\sin\theta\sin\frac{4\pi}{3},\cos\theta) & = (-\frac{1}{2}\sin\theta,-\frac{\sqrt{3}}{2}\sin\theta,\cos\theta) \\ \hat{n}_4 & = & (\sin\theta\cos 0,\sin\theta\sin 0,\cos\theta) & = (\sin\theta,0,\cos\theta) \end{array} \tag{22}$$

The symmetrical case corresponds to $\cos\theta = \frac{1}{3}$. The case $\theta = \frac{\pi}{2}$ is just the two-dimensional regular trine, while $\theta = \pi$ and $\theta = 0$ both give the straight line. None of the two latter configurations are informationally complete.
Proceeding as in the previous case, we consider the sum

$$\sum_{j=1}^{4} Y_{ll}(\hat{n}_j) \propto (-\frac{1}{2}+i\frac{\sqrt{3}}{2})^l + (-\frac{1}{2}-i\frac{\sqrt{3}}{2})^l + 1^l = \mathrm{e}^{il\alpha} + \mathrm{e}^{il\beta} + 1 \tag{23}$$

with

$$\begin{array}{c} \alpha \equiv \arctan -\sqrt{3} = \frac{\pi}{3} \\ \beta \equiv \arctan \sqrt{3} = \frac{4\pi}{3} \end{array}$$

This leads to the following condition for the Y_{ll}'s to sum to zero.

$$\cos l\alpha = \cos l\frac{\pi}{3} = -\frac{1}{2} \quad \Rightarrow \quad l=0, \quad l=2, \quad l \equiv 1 \pmod 2 \quad \wedge \quad l \not\equiv 0 \pmod 3 \tag{24}$$

Hence, the harmonics that can possibly contribute for this type of tetrahedra have l equal to zero, two or to an odd number that is not a multiple of three.
To find out if these values of l really do contribute, resulting, for any $l \neq 0, 1$ contributing, in the lack of a Gleason theorem, we proceed as in the previous case. Due to the ϕ dependence, the cases in which the sum

$$\sum_{j=1}^{4} Y_{lm}(\hat{n}_j)$$

will be non-zero occur for $m = 3, 6, 9$...and so on. For $l = 5$, $m = 3$ is the only allowed multiple of three, and for the values of θ that give $P_5^3(\cos\theta) = 0$ the $l = 5$ harmonic contributes, and we do not have a Gleason theorem. The zeros of P_5^3 are, apart from 0 and ± 1, $\pm\frac{1}{3}$ - the regular tetrahedron. Higher values of l either will be even or will allow at least two m-values that are multiples of three. Due to the lack of common zeroes of the associated Legendre polynomials for different m, we can deduce that only $l = 0, 2$ can contribute, so that we do get the standard quantum rule, except for the cases which give $P_5^3(\cos\theta) = 0$.
To summarize, we have found, assuming continuity, that a Gleason-type theorem can be proved for all types of four element POVMs in the two parameter families considered, with the exception of the regular tetrahedron (sic-POVM) and the square. The two ways of deforming a regular tetrahedron used are arguably the two most symmetric ways of introducing irregularity. Seeing as how it appears to be the high degree of symmetry that causes the proof to fail for the sic-POVM (that this might be the case was suggested by C. Fuchs, private communication) it is not likely that any other tetrahedra would exhibit a behaviour like that of the regular tetrahedron.

3 An incomplete Kochen-Specker colouring

It is worth noting that the proof of the Kochen-Specker theorem also simplifies if arbitrary POVMs are considered [7]. However, our present concern is quite different.
The first purpose of this section is to generalize to arbitrary dimension a truth value assignment, or colouring, originally proposed by Appleby in dimension three. Thereafter we will use two different measures of the effectivity of this colouring. Thereby, we will have derived a lower limit for the effectivity of a maximally effective Kochen-Specker colouring. The statement of the Kochen-Specker theorem being that this maximal effectivity does not equal one.
Let us first consider the three-dimensional case and then move on from there to higher dimensions. We are interested in assigning value 0 or 1 to vectors in $\mathscr{H}^3$ in such a way that no set of three mutually orthogonal vectors are all assigned the value 0, and no pair of orthogonal vectors both have the value 1. These conditions can be expressed as

$$g : S^2 \to \{0, 1\} \tag{25}$$

$$g(P_1) + g(P_2) + g(P_3) = 1 \tag{26}$$

for all sets of orthogonal vectors $\{P_1, P_2, P_3\}$, S^2 being the unit two-sphere. Letting white represent the value 0 and black the value 1, this problem can be translated into the problem of colouring S^2, in a way that satisfies the conditions just stated.
Any such assignment of truth values (probabilites from $\{0, 1\}$) to all vectors in the Hilbert space of some system would correspond to (the possibility of) the system having well-defined properties, existing independent of measurement. That is, for any possible observable the outcome of the corresponding measurement would be fully determined in advance. However, by the Kochen-Specker theorem, a complete such assigment of truth values is impossible. Hence, what we will try to do in the following is to assign probabilites from $\{0, 1\}$ according to (26) to some of the vectors in $\mathscr{H}$ - some vectors will necessarily remain uncoloured, by KS.
The way to go about this suggested by Appleby, is to start out by colouring the two polar caps defined by $|\tan\theta| < 1$ black, and the region around the equator bounded by $|\tan\theta| = \sqrt{2}$ white, where θ is the usual polar angle. This type of colouring is sketched in figure 1.

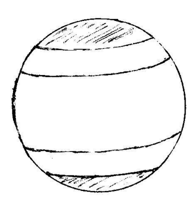

Figure 1: A possible (incomplete) KS colouring of the unit two-sphere.

These limits are derived as follows. The two polar caps are made small enough so that no two vectors in an orthogonal triple can simultaneously lie in the black region, which means that they will extend down to $\theta = \frac{\pi}{4}$. The white section around the equator is just wide enough so that not all three vectors can lie in it at the same time.
Already in four dimensions the contribution to the total area by the black cap is close to negligible. As we will see below it will reduce further with increasing dimension, which is why we in the following will primarily be interested in looking at the area taken up by the white section.
The fraction of the sphere in N dimensions that can be coloured white with the given restriction is

$$F = \frac{\int_{\arcsin\sqrt{\frac{N-1}{N}}}^{\frac{\pi}{2}} \sin^{N-2}\theta d\theta}{\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta d\theta} = 2\frac{\mathrm{vol}(S^{N-2})}{\mathrm{vol}(S^{N-1})}\int_{\arcsin\sqrt{\frac{N-1}{N}}}^{\frac{\pi}{2}} \sin^{N-2}\theta d\theta \tag{27}$$

where $\mathrm{vol}(S^d)$ denotes the surface area of the d-dimensional sphere. The intergral limits are derived using the expression

$$R_n = \sqrt{\frac{n}{n+1}} = \sqrt{\frac{N-1}{N}} \tag{28}$$

for the radius of the circumsphere of a regular n-simplex, where $n = N - 1$ is the dimension of the sphere in N dimensions.

As for the black area, B_N, it will in analogy with the $N = 3$ case be located around the poles of the sphere, with limiting angle $\frac{\pi}{4}$;

$$B_N = \mathrm{vol}(S^{N-2}) \int_0^{\frac{\pi}{4}} \sin^{N-2}\theta d\theta \tag{29}$$

What, one may ask, is the fraction of the sphere in N dimensions that can be coloured using this method in the limit $N \to \infty$? As can be seen from the expression

$$\mathrm{vol}(S^d) = \mathrm{vol}(S^{d-1}) \int_0^{\pi} \sin^{d-1}\theta d\theta \tag{30}$$

for high dimensions, the fraction of the area of the sphere that will lie around the poles is negligible, due to the increasingly sharp peak around $\theta = \frac{\pi}{2}$ of the sine function when raised to a large number. Thus the fraction of the surface area taken up by the black section will be very small.
To determine the fraction of the sphere taken up by the white section requires a bit more careful analysis. We will need to evaluate the expression

$$\lim_{N\to\infty} 2\frac{\mathrm{vol}(S^{N-2})}{\mathrm{vol}(S^{N-1})} \int_{\arcsin\sqrt{\frac{N-1}{N}}}^{\frac{\pi}{2}} \sin^{N-2}\theta d\theta \tag{31}$$

Using the known formula for $\mathrm{vol}(S^d)$ we find that

$$\frac{\mathrm{vol}(S^{N-2})}{\mathrm{vol}(S^{N-1})} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})} \tag{32}$$

This tends to $\frac{\sqrt{N}}{\sqrt{2}}$ as $N \to \infty$.
Next, let us take a look at the behaviour of the integral

$$\int_{\arcsin\sqrt{\frac{N-1}{N}}}^{\frac{\pi}{2}} \sin^{N-2}\theta d\theta = \int_0^{\arccos\sqrt{\frac{N-1}{N}}} \cos^{N-2}\theta d\theta \tag{33}$$

in the limit of large N. The second form is convenient because all expansions can be done around zero.
In the limit of large N we can use

$$\arccos\sqrt{\frac{N-1}{N}} = \frac{1}{\sqrt{N}} + O(\frac{1}{N^{\frac{3}{2}}}) \tag{34}$$

Expanding $\cos t$ around $t = 0$ and using the regular binomial expansion and the fact that when N is large $N - 2$ can be approximated with N,

$$\lim_{N\to\infty} \cos^{N-2}\theta = (1 - \frac{\theta^2}{2})^N + h(\theta, N) = h(\theta, N) + 1 - N\frac{\theta^2}{2} + \frac{N^2}{2!}\frac{\theta^4}{4} - \frac{N^3}{3!}\frac{\theta^6}{8} + ... \tag{35}$$

where $h(\theta, N)$ is a function such that

$$\lim_{N\to\infty} \sqrt{N} \int_0^{\arccos\sqrt{\frac{N-1}{N}}} h(\theta, N) = 0 \tag{36}$$

Integrating term by term and using the cosine expansion and equation (34) for the expansion of arccosine, we get

$$\lim_{N\to\infty}\int_0^{\arccos\sqrt{\frac{N-1}{N}}}\cos^{N-2}\theta d\theta = \frac{1}{\sqrt{N}}\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{1}{k!}\frac{(-1)^k}{(2k+1)} = \frac{\pi}{\sqrt{2N}}\mathrm{erf}(\frac{1}{\sqrt{2}}) \tag{37}$$

with erf the statistic-probabilistic error function;

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}dt \tag{38}$$

Putting all of this together, we have the result

$$\lim_{N\to\infty} 2\frac{\mathrm{vol}(S^{n-2})}{\mathrm{vol}(S^{n-1})}\int_{\arcsin\sqrt{\frac{N-1}{N}}}^{\frac{\pi}{2}}\sin^{N-2}\theta d\theta = \mathrm{erf}(\frac{1}{\sqrt{2}}) \approx 0.68 \tag{39}$$

So, approaching the limit of an infinite number of dimensions of the Hilbert space $\mathscr{H}$ in which our projective measurements are conducted, binary probabilities (corresponding to well-defined, non-contextual properties of the system with available states in $\mathscr{H}$) can be assigned to approximately 68% of the vectors in $\mathscr{H}$.
The behaviour of the percentage as a function of dimension is given in figure 2. The results are $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}=87\%$ of all vectors for n=2, 79% for $n=3$, 74% for $n=4$ and 71% for $n=5$. The integer giving the least percentage is $N=12$; about 66.76%.

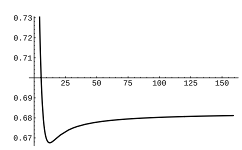

Figure 2: Percentage of the sphere in N dimensions that is colourable using the above method, as a function of N.

What has been derived above is a lower limit for the area of the sphere that is KS colourable in arbitrary dimension. The possibility remains, however, that a maximally effective colouring could cover a much larger area - possibly, in fact, as much as 99% of the sphere in $\mathbb{R}^3$. The physically relevant question is, arguably, not how large a fraction of all states can be assigned probabilities 1 or 0, but rather what percentage of all complete orthogonal bases (measurements)can have all their basis vectors assigned binary probabilities in a consistent way. We will answer this question specifically for the Appleby colouring in three and four dimensions.
Let us first consider the colouring of the two-sphere proposed above - a black cap and a white equatorial belt covering in total 87% of the sphere - and make use of the regular measure on $\mathbb{R}^3$ to compare the number of properly coloured bases consisting of vectors from these sections with the total number of ordered orthonormal triples in $\mathbb{R}^3$.
In a properly coloured base exactly one vector has to be black, so one of the three vectors in an orthogonal triple has to be chosen to lie on one of the black caps. The remaining two orthogonal vectors can then be chosen from a great circle orthogonal to the first vector - the question is how large a fraction of this great circle will lie within the white section and also how the second vector (which, of course, completely determines the third basis vector up to a sign) can be chosen so that the third vector will also be contained within the white section.

Figure 3 depicts the plane of the great circle orthogonal to the first (black) vector on which the remaining two vectors in the orthogonal triple will have to lie. The circle segment bounding the

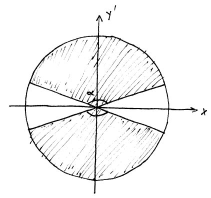

Figure 3: A cut through the plane of the great circle orthogonal to the vector chosen to lie on the black cap.

striped area is the cut between the white belt and the orthogonal great circle. For any choice of second vector from this section, the third vector will be fully determined (up to a sign). Hence, we cannot choose our second vector in a satisfactorily coloured triple from any part of the circle-belt overlap in figure 3, but only from the sectors that will result in the third vector lying in the white belt as well. Given a second vector, the third is obtained by rotation in the great circle plane by an angle of $\frac{\pi}{2}$. The allowed choices for second vector are then the points such that the points corresponding to a $\frac{\pi}{2}$ rotation of these points are also white. This set of points is just the overlap between the white (striped) sector in figure 3 and the same sector rotated by $\frac{\pi}{2}$, as illustrated in figure 4, an overlap that can be shown to always be non-empty. Hence, what we will need to find is the total angle taken up by the striped section in figure 4 - this will be denoted by β. It is clear that this β can be expressed in terms of the α of figure 3 as

$$\beta = 4\alpha - 2\pi \tag{40}$$

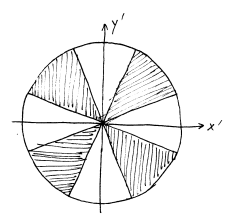

Figure 4: Overlap between the white section of the great circle, and its rotation by $\frac{\pi}{2}$.

The angle α, in turn, can be expressed in terms of the regular polar angle θ that specifies our choice of black vector using the following procedure.
First, consider the plane spanned by the vector chosen to lie in the black section, call it z', and a vector y' in the plane orthogonal to z'; $\{x, y, z\}$ is a reference coordinate system as shown. The vector x' orthogonal to y' and z' is chosen so that its z component equals zero. From figure 5 it is clear that

$$z = 0x' + \sin\theta y' + \cos\theta z' \tag{41}$$

Meanwhile, as can be seen from figure 6, a vector v lying just on the boundary of the white belt can be expressed in terms of y' and x' as

$$v = \cos\alpha' x' + \sin\alpha' y' \tag{42}$$

with $\alpha' = \frac{\alpha}{2}$, its z component being equal to zero. We also know that the z component of our vector v is just h, with $h = \frac{1}{\sqrt{3}}$ according to our earlier deliberations Taken together, this gives

$$v \cdot z = h = (\cos\alpha' x' + \sin\alpha' y') \cdot z = \sin\alpha' y' \cdot z = \sin\alpha' \sin\theta \tag{43}$$

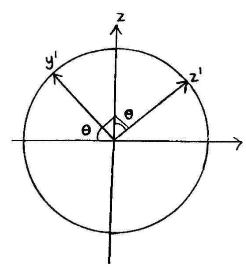

Figure 5: The vector y' will make an angle $\pi - \theta$ with the z axis.

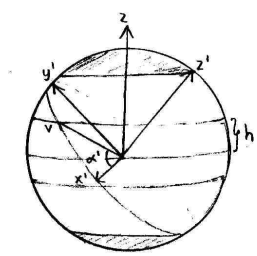

Figure 6: Coordinates x' and y' are introduced in the plane of the great circle orthogonal to the vector z'.

so that

$$\alpha = 2 \arcsin \frac{h}{\sin \theta} \tag{44}$$

and

$$\beta = 8 \arcsin \frac{h}{\sin \theta} - 2\pi \tag{45}$$

When $\theta < \arcsin \frac{1}{\sqrt{3}}$ expression (44) for α will not be defined; for those angles all of the vectors orthogonal to the black section vector defined by the angle θ will lie within the white section.
This enables us to express the fraction of the orthogonal great circle corresponding to every choice of vector z' in terms of the angle θ, making possible integration over all values of θ and thereby the comparison we have in mind.
So, the integrals we will want to evaluate are

$$I = 2\pi \int_0^{\arcsin \frac{1}{\sqrt{3}}} \sin \theta d\theta + \int_{\arcsin \frac{1}{\sqrt{3}}}^{\frac{\pi}{4}} (8 \arcsin \frac{h}{\sin \theta} - 2\pi) \sin \theta d\theta \tag{46}$$

the value of which turns out to be 1.4572.
This, multiplied by a combinatorial factor of three because what we considered the first vector could as well have been the second or third, should be compared to the value of the integral

$$2\pi \int_0^{\frac{\pi}{2}} \sin \theta d\theta = 2\pi \tag{47}$$

- the result is that approximately 69% of all possible ordered bases in $\mathbb{R}^3$ can be satisfactorily KS-coloured using the given construction.

The above considerations for the three-dimensional case can with some modifications be applied also in four dimensions. Introducing spherical coordinates $\{\phi, \theta_1, \theta_2\}$ on the three-sphere, we will start out by finding the intersectional area of the orthgonal two-sphere and the white 'belt'. Let z' denote the black vector, let z be a reference coordinate, and let y' be a vector on the orthogonal two-sphere as in figure 7.

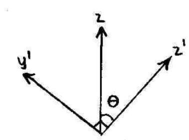

Figure 7: y' lies on the two-sphere orthogonal to the vector z'.

The white section is the set of vectors

$$\{u : \quad | u \cdot z | \leq A\}, \quad A = \frac{1}{2} \tag{48}$$

For any vector u in this set we have that

$$u \cdot z' = 0 \tag{49}$$

Now, let's make the ansatz

$$y' = az + bz' \tag{50}$$

Normalization together with condition (49) then gives

$$a = \frac{1}{\sin\theta_2}, \quad b = -\frac{\cos\theta_2}{\sin\theta_2} \tag{51}$$

Also,

$$u \cdot z' = 0 \Rightarrow u \cdot y' = u \cdot (az + bz') = au \cdot z \tag{52}$$

So, using (48), the belt on the orthogonal two-sphere will be the set of vectors

$$\{v : \quad | v | \leq B\}, \quad B = aA = \frac{1}{2\sin\theta_2} \tag{53}$$

For $0 \leq \theta_2 \leq \arcsin\frac{1}{2}$ the orthogonal two-sphere will lie entirely within the white section.
This intersection between the orthogonal two-sphere and the white section on the three-sphere can now be treated in analogue with the previuos case. Given a black first vector, when placing the second vector in the white section, the segment of the great circle orthogonal to this second vector on which we can choose the third in order for the fourth to lie in the white section is given by

$$\gamma = 8 \arcsin\frac{B}{\sin\theta_1} - 2\pi \tag{54}$$

Also in analogy with the previous case, all of the orthogonal great circle will be white for $\arccos B \leq \theta_1 \leq \arcsin B$. To summarize, we have integration over the angle θ_2 which runs between 0 and $\frac{\pi}{2}$, covering the black cap, and the possibilities available for choosing the remaining three vectors are governed by a function of θ_2, obtained from an integration over the angle θ_1 between $\arccos B$ and $\frac{\pi}{2}$, that is, over the white section of the two-sphere orthogonal to the first vector specified by

θ_2, B being a function of θ_2.
To make all of this explicit, we have the following integrals

$$I = 2\pi \int_{\arccos B}^{\arcsin B} \sin\theta_1 d\theta_1 + \int_{\arcsin B}^{\frac{\pi}{2}} (8 \arcsin \frac{B}{\sin\theta_1} - 2\pi) \sin\theta_1 d\theta_1 \tag{55}$$

and, finally

$$4\pi \int_0^{\arcsin\frac{1}{2}} \sin^2\theta_2 d\theta_2 + \int_{\arcsin\frac{1}{2}}^{\frac{\pi}{4}} I \sin^2\theta_2 d\theta_2 \tag{56}$$

The result when comparing this, multiplied by an overall combinatorial factor of four, to the value of the expression

$$4\pi \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \tag{57}$$

is that 32% of the ordered orthogonal triples in $\mathbb{R}^4$ are properly coloured using the chosen method.

4 Conclusions

In the case of two-dimensional quantum systems, qubits, the sic-POVM does not yield the Gleason result, as shown also by Caves et al. By proving the validity of a Gleason-type theorem for two classes of irregular tetrahedrons, under the assumption of continuity, we have made likely that for all other four element POVMs, with the exception of the square configuration, the quantum rule holds.
Generalizing a method of colouring proposed by Appleby in three dimensions, we have also found a lower bound on the area of the D-sphere that can be KS coloured, but we are still ignorant as to a sharp upper bound.
In three and four dimensions, we have calculated how many of all possible bases the coloured area corresponds to. In order to make the lower bound mentioned above more physically interesting, one would need to answer this question in arbitrary dimensions - something that appears to be non-trivial.
The Appleby colouring has the advantage that it generalizes easily to higher dimensions. As for a maximally effective colouring, there are no arguments to support that this would be the case. In particular, it is in no way obvious that the same method of colouring would be maximal in different dimensions.

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